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MULTIPLE EXPANSIONS OF REAL NUMBERS WITH DIGITS SET $\{0, 1, q\}$

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Dedicated to Michel Dekking on the occasion of his 70th birthday

ABSTRACT. For $q > 1$ we consider expansions in base q over the alphabet $\{0, 1, q\}$. Let $\mathcal{U}_q$ be the set of x which have a unique q-expansions. For $k = 2, 3, \cdots, \aleph_0$ let $\mathcal{B}_k$ be the set of bases q for which there exists x having k different q-expansions, and for $q \in \mathcal{B}_k$ let $\mathcal{U}_q^{(k)}$ be the set of all such x's which have k different q-expansions. In this paper we show that

$$\mathcal{B}_{\aleph_0} = [2, \infty), \quad \mathcal{B}_k = (q_c, \infty) \quad \text{for any} \quad k \geq 2,$$

where $q_c \approx 2.32472$ is the appropriate root of $x^3 - 3x^2 + 2x - 1 = 0$. Moreover, we show that for any positive integer $k \geq 2$ and any $q \in \mathcal{B}_k$ the Hausdorff dimensions of $\mathcal{U}_q^{(k)}$ and $\mathcal{U}_q$ are the same, i.e.,

$$\dim_H \mathcal{U}_q^{(k)} = \dim_H \mathcal{U}_q \quad \text{for any} \quad k \geq 2.$$

Finally, we conclude that the set of x having a continuum of q-expansions has full Hausdorff dimension.

1. Introduction

Expansions in non-integer bases were pioneered by Rényi [17] and Parry [15]. It is well-known that typically a real number has a continuum of expansions (cf. [18, 2]). However, there still exist reals having a unique expansion (cf. [4, 9, 12]). Recently, de Vries and Komornik [3] investigated the topological properties of unique expansions. Komornik et al. [11] considered the Hausdorff dimension of unique expansions,

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and conclude that the dimension function behaves like a devil's staircase. Interestingly, for $k = 2, 3, \cdots$ or $\aleph_0$ it was first discovered by Erdős et al. [5, 6] that there exists x having k different expansions. For more information on expansions in non-integer bases we refer to [20, 1, 22], and surveys [19, 10].

In this paper we consider expansions with digits set $\{0, 1, q\}$. Given $q > 1$, an infinite sequence (d_i) is called a *q-expansion* of x if

$$x = ((d_i))_q := \sum_{i=1}^{\infty} \frac{d_i}{q^i}, \quad d_i \in \{0, 1, q\}, \quad i \geq 1.$$

Here we point out that the expansion is over the *alphabet* $\{0, 1, q\}$ which depends on the base q.

For $q > 1$ let E_q be the set of x which have a q-expansion. Then E_q is the attractor of the *iterated function system* (cf. [7])

$$\phi_d(x) = \frac{x + d}{q}, \quad d \in \{0, 1, q\},$$

i.e., E_q is the non-empty compact set satisfying $E_q = \bigcup_{d\in\{0,1,q\}} \phi_d(E_q)$.

The set E_q is a *self-similar set* with overlaps, and it attracts great attention since the work of Nagi and Wang [14] for giving an explicit formulae for the Hausdorff dimension of E_q:

$$\dim_H E_q = \frac{\log q^*}{\log q} \quad \text{for any} \quad q > q^*, \tag{1.1}$$

where $q^* = (3 + \sqrt{5})/2$. Moreover, Yao and Li [21] considered all of its generating iterated function systems of the set E_q. Recently, Zou et al. [23] considered the set of points in E_q which have a unique q-expansion. Then it is natural to ask what can we say for points in E_q having multiple q-expansions?

For $k = 1, 2, \cdots, \aleph_0$ or $2^{\aleph_0}$, let $\mathcal{B}_k$ be the set of bases $q > 1$ such that there exists $x \in E_q$ having k different q-expansions. Accordingly, for $q \in \mathcal{B}_k$ let $\mathcal{U}_q^{(k)}$ be the set of $x \in E_q$ having k different q-expansions. Then $\mathcal{B}_k = \left\{q > 1 : \mathcal{U}_q^{(k)} \neq \emptyset\right\}$, and for $q \in \mathcal{B}_k$

$$\mathcal{U}_q^{(k)} = \{x \in E_q : x \text{ has } k \text{ different } q\text{-expansions}\}.$$

For simplicity, we write $\mathcal{U}_q := \mathcal{U}_q^{(1)}$ for the set of $x \in E_q$ having a unique q-expansion, and denote by $\mathcal{U}_q'$ the set of corresponding expansions.

First we consider the set $\mathcal{B}_k$ for $k = 1, 2, \cdots, \aleph_0$ or $2^{\aleph_0}$. Clearly, when $k = 1$ we have $\mathcal{B}_1 = (1, \infty)$, since 0 always has a unique q-expansion for any $q > 1$.

When $k = 2, 3, \cdots, \aleph_0$ or $2^{\aleph_0}$ we have the following theorem.

Theorem 1.1. *Let $q_c \approx 2.32472$ be the appropriate root of $x^3 - 3x^2 + 2x - 1 = 0$. Then*

$$\mathcal{B}_{2^{\aleph_0}} = (1, \infty), \quad \mathcal{B}_{\aleph_0} = [2, \infty), \quad \mathcal{B}_k = (q_c, \infty) \quad \textit{for any} \quad k \geq 2.$$

In terms of Theorem 1.1 it follows that for $q \in [2, q_c]$ any $x \in E_q$ can only have a unique q-expansion, countably infinitely many q-expansions, or a continuum of q-expansions.

For $k \geq 1$ and $q \in \mathcal{B}_k$ we consider the set $\mathcal{U}_q^{(k)}$. When $k = 1$, the following theorem for the *univoque set* $\mathcal{U}_q = \mathcal{U}_q^{(1)}$ was shown in [23].

Theorem 1.2 ([23]).
- *If $q \in (1, q_c]$, then $\mathcal{U}_q = \{0, q/(q-1)\}$;*
- *If $q \in (q_c, q^*)$, then $\mathcal{U}_q$ contains a continuum of points;*
- *If $q \in [q^*, \infty)$, then $\dim_H \mathcal{U}_q = \log q_c / \log q$.*

In the following theorem we show that the Hausdorff dimensions of $\mathcal{U}_q^{(k)}$ are the same for any integer $k \geq 1$.

Theorem 1.3. *For any integer $k \geq 2$ and any $q \in \mathcal{B}_k$ we have*

$$\dim_H \mathcal{U}_q^{(k)} = \dim_H \mathcal{U}_q.$$

Moreover, $\dim_H \mathcal{U}_q > 0$ if and only if $q > q_c$.

In terms of Theorem 1.3 it follows that q_c is indeed the *critical base* in the sense that $\mathcal{U}_q^{(k)}$ has positive Hausdorff dimension if $q > q_c$, while $\mathcal{U}_q^{(k)}$ has zero Hausdorff dimension if $q \leq q_c$. In fact, by Theorems 1.1 and 1.2 it follows that for $q \leq q_c$ the set $\mathcal{U}_q = \{0, q/(q-1)\}$ and $\mathcal{U}_q^{(k)} = \emptyset$ for any integer $k \geq 2$.

In the following theorem we consider $\mathcal{U}_q^{(\aleph_0)}$ and $\mathcal{U}_q^{(2^{\aleph_0})}$.

Theorem 1.4.
- *Let $q \in \mathcal{B}_{\aleph_0} \setminus (q_c, q^*)$. Then $\mathcal{U}_q^{(\aleph_0)}$ contains countably infinitely many points;*
- *Let $q > 1$. Then $\mathcal{U}_q^{(2^{\aleph_0})}$ has full Hausdorff dimension, i.e.,*

$$\dim_H \mathcal{U}_q^{(2^{\aleph_0})} = \dim_H E_q.$$

Remark 1.5. In fact, we show in Lemma 5.5 that the Hausdorff measures of $\mathcal{U}_q^{(2^{\aleph_0})}$ and E_q are the same for any $q > 1$, i.e.,

$$\mathcal{H}^s(\mathcal{U}_q^{(2^{\aleph_0})}) = \mathcal{H}^s(E_q) \in (0, \infty),$$

where $s = \dim_H E_q$.

The rest of the paper is arranged in the following way. In Section 2 we recall some properties of unique q-expansions. The proof of Theorem 1.1 for the sets $\mathcal{B}_k$ will be presented in Section 3, and the proofs of Theorems 1.3 and 1.4 for the sets $\mathcal{U}_q^{(k)}$ will be given in Sections 4 and 5, respectively. Finally, in Section 6 we give some examples and end the paper with some questions.

2. UNIQUE EXPANSIONS

In this section we recall some properties of the univoque set $\mathcal{U}_q$. Recall that

$$q_c \approx 2.32472, \quad q^* = \frac{3+\sqrt{5}}{2}.$$

Here q_c is the appropriate root of the equation $x^3 - 3x^2 + 2x - 1 = 0$. Then for $q \in (1, q^*]$ the attractor $E_q = [0, q/(q-1)]$ is an interval. However, for $q > q^*$ the attractor E_q is a Cantor set which contains neither interior nor isolated points.

The following characterization of the univoque set $\mathcal{U}_q$ for $q > q^*$ was established in [23, Lemma 3.1].

Lemma 2.1. *Let $q > q^*$. Then $(d_i) \in \mathcal{U}_q'$ if and only if*

$$\begin{cases} (d_{n+i}) < q0^\infty & \textit{if} \quad d_n = 0, \\ (d_{n+i}) > 1^\infty & \textit{if} \quad d_n = 1. \end{cases}$$

In the following we consider unique q-expansions with $q \le q^*$. For $q \in (1, q^*]$ we denote by

$$\alpha(q) = (\alpha_i(q))$$

the *quasi-greedy* q-expansion of $q-1$, i.e., the lexicographical largest infinite q-expansion of $q-1$. Here an expansion (d_i) is called *infinite* if $d_i \neq 0$ for infinitely many indices $i \ge 1$.

In terms of Theorem 1.2 it is interesting to consider the set $\mathcal{U}_q'$ of unique expansions for $q \in (q_c, q^*]$. The following lemma was obtained in [23, Lemmas 3.1 and 3.2].

Lemma 2.2. *Let $q \in (q_c, q^*]$. Then*

$$A_q \subseteq \mathcal{U}_q' \subseteq B_q,$$

where A_q is the set of sequences $(d_i) \in \{0, 1, q\}^\infty$ satisfying

$$\begin{cases} (d_{n+i}) < 1\alpha(q) & \text{if} \quad d_n = 0, \\ 1^\infty < (d_{n+i}) < \alpha(q) & \text{if} \quad d_n = 1, \\ (d_{n+i}) > 0q^\infty & \text{if} \quad d_n = q, \end{cases} \tag{2.1}$$

and B_q is the set of sequences $(d_i) \in \{0, 1, q\}^\infty$ satisfying the first two inequalities in (2.1).

For $q > 1$ let $\Phi : \{0, 1, q\}^\infty \to \{0, 1, 2\}^\infty$ be defined by

$$\Phi((d_i)) = (d_i'),$$

where $d_i' = d_i$ if $d_i \in \{0, 1\}$, and $d_i' = 2$ if $d_i = q$. Clearly, the map Φ is continuous and bijective.

The following monotonicity of $\Phi(\alpha(q))$ was given in [23, Lemma 3.2].

Lemma 2.3. *The map $q \to \Phi(\alpha(q))$ is strictly increasing in $(1, q^*]$.*

3. Proof of Theorem 1.1

In this section we will investigate the set $\mathcal{B}_k$ of bases $q > 1$ in which there exists $x \in E_q$ having k different q-expansions. When $k = 1$ it is obviously that $\mathcal{B}_1 = (1, \infty)$ because $0 \in E_q$ always has a unique q-expansion 0^∞ for any $q > 1$. In the following we consider $\mathcal{B}_k$ for $k = 2, 3, \cdots, \aleph_0$ or $2^{\aleph_0}$.

The following lemma was established in [23, Theorem 4.1] and [8, Theorem 1.1].

Lemma 3.1. *Let $q \in (1, 2)$. Then any $x \in E_q$ has either a unique q-expansion, or a continuum of q-expansions.*

Moreover, for $q = 2$ any $x \in E_q$ can only have a unique q-expansion, countably infinitely many q-expansions, or a continuum of q-expansions.

For $q>1$ we recall that $\phi_d(x)=(x+d)/q, d=0,1,q$. Let

$$S_q:=(\phi_0(E_q)\cap\phi_1(E_q))\cup(\phi_1(E_q)\cap\phi_q(E_q))\,. \tag{3.1}$$

Here S_q is called the *switch region*, since any $x\in S_q$ has at least two q-expansions. Clearly, any $x\in\phi_0(E_q)\cap\phi_1(E_q)$ has at least two q-expansions: one beginning with the word 0 and one beginning with the word 1. Accordingly, any $x\in\phi_1(E_q)\cap\phi_q(E_q)$ also has at least two q-expansions: one starting at the word 1 and one starting at the word q. We point out that the union in (3.1) is disjoint if $q>2$. In fact, for $q>q^*$ the intersection $\phi_1(E_q)\cap\phi_q(E_q)=\emptyset$.

For $x\in E_q$ let $\Sigma(x)$ be the set of all q-expansions of x, i.e.,

$$\Sigma(x):=\left\{(d_i)\in\{0,1,q\}^\infty:((d_i))_q=x\right\},$$

and denote by $|\Sigma(x)|$ its cardinality.

We recall from [1] that a point $x\in S_q$ is called a *q-null infinite point* if x has an expansion (d_i) such that whenever

$$x_n:=(d_{n+1}d_{n+2}\cdots)_q\in S_q,$$

one of the following quantities is infinity, and another two are finite:

$$\left|\Sigma(\phi_0^{-1}(x_n))\right|,\quad\left|\Sigma(\phi_1^{-1}(x_n))\right|\quad\text{and}\quad\left|\Sigma(\phi_q^{-1}(x_n))\right|.$$

Clearly, any q-null infinite point has countably infinitely many q-expansions. In order to investigate $\mathcal{B}_{\aleph_0}$ we need the following relationship between $\mathcal{B}_{\aleph_0}$ and q-null infinite points which was established in [1] (see also, [22]).

Lemma 3.2. *$q\in\mathcal{B}_{\aleph_0}$ if and only if S_q contains a q-null infinite point.*

First we consider the set $\mathcal{B}_{\aleph_0}$.

Lemma 3.3. $\mathcal{B}_{\aleph_0}=[2,\infty)$.

Proof. By Lemma 3.1 we have $\mathcal{B}_{\aleph_0}\subseteq[2,\infty)$ and $2\in\mathcal{B}_{\aleph_0}$. So, it suffices to prove $(2,\infty)\subseteq\mathcal{B}_{\aleph_0}$.

Take $q\in(2,\infty)$. Note that $0=(0^\infty)_q$ and $q/(q-1)\in(q^\infty)_q$ belong to $\mathcal{U}_q$. We claim that

$$x=(0q^\infty)_q$$

is a q-null infinite point.

By the words substitution $10 \sim 0q$ it follows that all expansions $1^k 0 q^\infty, k \geq 0$, are q-expansions of x, i.e.,

$$\bigcup_{k=0}^{\infty} \{1^k 0 q^\infty\} \subseteq \Sigma(x).$$

This implies that $|\Sigma(x)| = \infty$.

Furthermore, since $q > 2$, the union in (3.1) is disjoint. This implies

$$x = (0q^\infty)_q = (10q^\infty)_q \in \phi_0(E_q) \cap \phi_1(E_q) \setminus \phi_q(E_q).$$

Then $\phi_0^{-1}(x) = (q^\infty)_q \in \mathcal{U}_q$, $\phi_1^{-1}(x) = x$ and $\phi_q^{-1}(x) \notin E_q$, i.e.,

$$|\Sigma(\phi_0^{-1}(x))| = 1, \quad |\Sigma(\phi_1^{-1}(x))| = \infty, \quad |\Sigma(\phi_q^{-1}(x))| = 0.$$

By iteration it follows that x is a q-null infinite point. Hence, by Lemma 3.2 it yields that $q \in \mathcal{B}_{\aleph_0}$, and therefore $(2, \infty) \subseteq \mathcal{B}_{\aleph_0}$. □

In the following we will consider $\mathcal{B}_k$. By Lemma 3.1 it follows that $\mathcal{B}_k \subseteq (2, \infty)$ for any $k \geq 2$. First we consider $k = 2$. In the following lemma we give a characterization of the set $\mathcal{B}_2$.

Lemma 3.4. *Let $q > 2$. Then $q \in \mathcal{B}_2$ if and only if there exist $(a_i), (b_i) \in \mathcal{U}_q'$ such that*

$$(1(a_i))_q = (0(b_i))_q,$$

or there exist $(c_i), (d_i) \in \mathcal{U}_q'$ such that

$$(1(c_i))_q = (q(d_i))_q.$$

Proof. First we prove the necessity. Take $q \in \mathcal{B}_2$. Suppose $x \in E_q$ has two different q-expansions, say

$$((a_i))_q = x = ((b_i))_q.$$

Then there exists a least integer $k \geq 1$ such that $a_k \neq b_k$. Then

$$(3.2) \quad (a_k a_{k+1} \cdots)_q = (b_k b_{k+1} \cdots)_q \in S_q, \quad \text{and} \quad (a_{k+i})_q, (b_{k+i})_q \in \mathcal{U}_q.$$

Since $q > 2$, it gives that the union in (3.1) is disjoint. Then the necessity follows by (3.2).

Now we turn to prove the sufficiency. Without loss of generality we assume $(1(a_i))_q = (0(b_i))_q$ with $(a_i), (b_i) \in \mathcal{U}'_q$. Note by $q > 2$ that the union in (3.1) is disjoint. Then

$$x = (1(a_i))_q = (0(b_i)) \in \phi_0(E_q) \cap \phi_1(E_q) \setminus \phi_q(E_q).$$

This implies that x has two different q-expansions, i.e., $q \in \mathcal{B}_2$. $\square$

Recall that $q_c \approx 2.32472$ is the appropriate root of $x^3 - 3x^2 + 2x - 1 = 0$, and $q^* = (3 + \sqrt{5})/2$. By a direct computation one can verify that

$$\alpha(q_c) = q_c 1^\infty, \quad \alpha(q^*) = (q^*)^\infty. \tag{3.3}$$

In the following lemma we consider the set $\mathcal{B}_2$.

Lemma 3.5. *$\mathcal{B}_2 = (q_c, \infty)$.*

Proof. First we show that $\mathcal{B}_2 \subseteq (q_c, \infty)$. By Lemma 3.1 it suffices to prove that $(2, q_c]$ is not contained in $\mathcal{B}_2$. Take $q \in (2, q_c]$. Then by Theorem 1.2 it gives that $\mathcal{U}_q = \{(0^\infty)_q, (q^\infty)_q\}$. In terms of Lemma 3.4 it follows that if $q \in \mathcal{B}_2 \cap (2, q_c]$ then q must satisfies one of the following equations

$$(10^\infty)_q = (0q^\infty)_q \quad \text{or} \quad (1q^\infty)_q = (q0^\infty)_q.$$

This is impossible since neither equation has a solution in $(2, q_c]$. Hence, $\mathcal{B}_2 \subseteq (q_c, \infty)$.

Now we turn to prove $(q_c, \infty) \subseteq \mathcal{B}_2$. In terms of Lemmas 2.1 and 3.4 one can verify that for any $q > q^*$ the number

$$x = (0q0^\infty)_q = (10^\infty)_q$$

has two different q-expansions. This implies that $(q^*, \infty) \subseteq \mathcal{B}_2$.

In the following it suffices to prove $(q_c, q^*] \subset \mathcal{B}_2$. Take $q \in (q_c, q^*]$. Note by (3.3) that $\alpha(q_c) = q_c 1^\infty$ and $\alpha(q^*) = (q^*)^\infty$. Then by Lemma 2.3 there exists a large integer m such that

$$\alpha(q) > q1^m q0^\infty.$$

Hence, by Lemmas 2.2 and 3.4 one can verify that

$$y = (0q(1^{m+1}q)^\infty)_q = (10(1^{m+1}q)^\infty)_q$$

has two different q-expansions. This implies that $(q_c, q^*] \subseteq \mathcal{B}_2$, and completes the proof. $\square$

Lemma 3.6. *$\mathcal{B}_k = (q_c, \infty)$ for any $k \geq 3$.*

Proof. First we prove $\mathcal{B}_k \subseteq \mathcal{B}_2$ for any $k \geq 3$. By Lemma 3.1 it follows that $\mathcal{B}_k \subseteq (2, \infty)$. Take $q \in \mathcal{B}_k$ with $k \geq 3$. Suppose $x \in E_q$ has k different q-expansions. Since $q > 2$, the union in (3.1) is disjoint. This implies that there exists a word $d_1 \cdots d_n$ such that

$$\phi_{d_1}^{-1} \circ \cdots \circ \phi_{d_n}^{-1}(x)$$

has two different q-expansions, i.e., $q \in \mathcal{B}_2$. Hence, $\mathcal{B}_k \subseteq \mathcal{B}_2$ for any $k \geq 3$.

Now we turn to prove $\mathcal{B}_2 \subseteq \mathcal{B}_k$ for any $k \geq 3$. In terms of Lemma 3.5 it suffices to prove

$$(q_c, \infty) \subseteq \mathcal{B}_k.$$

First we prove $(q^*, \infty) \subseteq \mathcal{B}_k$. Take $q \in (q^*, \infty)$. We only need to show that for any $k \geq 1$,

$$x_k = (0q^{k-1}(1q)^\infty)_q$$

has k different q-expansions. We will prove this by induction on k.

For $k = 1$ one can easily check by using Lemma 2.1 that $x_1 = (0(1q)^\infty)_q \in \mathcal{U}_q$. Suppose x_k has exactly k-different q-expansions. Now we consider x_{k+1}, which can be written as

$$x_{k+1} = (0q^k(1q)^\infty)_q = (10q^{k-1}(1q)^\infty)_q.$$

By Lemma 2.1 we have $q^k(1q)^\infty \in \mathcal{U}_q'$. Moreover, by the induction hypothesis $(0q^{k-1}(1q)^\infty)_q = x_k$ has exactly k different q-expansions. Then x_{k+1} has at least $k+1$ different q-expansions. On the other hand, note by $q > q^* > 2$ that the union in (3.1) is disjoint. Then

$$x_{k+1} \in \phi_0(E_q) \cap \phi_1(E_q) \setminus \phi_q(E_q).$$

This implies that x_{k+1} indeed has $k+1$ different q-expansions.

In the following we prove $(q_c, q^*] \subseteq \mathcal{B}_k$. Take $q \in (q_c, q^*]$. Then by (3.3) and Lemma 2.3 there exists a sufficiently large integer $m \geq 1$ such that

$$\alpha(q) > q1^m q0^\infty. \tag{3.4}$$

We will finish the proof by inductively showing that

$$y_k = (0q^{k-1}(1^{m+1}q)^\infty)_q$$

has k different q-expansions.

If $k = 1$, then by using (3.4) in Lemma 2.2 it gives that $y_1 = (0(1^{m+1}q)^\infty)_q$ has a unique q-expansion. Suppose y_k has exactly k-different q-expansions. Now we consider y_{k+1}. Clearly,

$$y_{k+1} = (10q^{k-1}(1^{m+1}q)^\infty)_q = (0q^k(1^{m+1}q)^\infty)_q.$$

By (3.4) and Lemma 2.2 it yields that $q^k(1^{m+1}q)^\infty \in \mathcal{U}'_q$. By induction we know that $(0q^{k-1}(1^{m+1}q)^\infty)_q = y_k$ has exactly k different q-expansions. This implies that y_{k+1} has at least $k+1$ different q-expansions. On the other hand, note that $q > q_c > 2$, and therefore the union in (3.1) is disjoint. So,

$$y_{k+1} \in \phi_0(E_q) \cap \phi_1(E_q) \setminus \phi_q(E_q),$$

which implies that y_{k+1} indeed has $k+1$ different q-expansions. □

Proof of Theorem 1.1. In terms of Lemmas 3.3, 3.5 and 3.6 it suffices to prove $\mathcal{B}_{2^{\aleph_0}} = (1, \infty)$. This can be verified by observing that

$$x = ((100)^\infty)_q \in \mathcal{U}_q^{(2^{\aleph_0})}$$

for any $q > 1$. Because by the word substitution $10 \sim 0q$ one can show that x indeed has a continuum of different q-expansions.

□

4. Proof of Theorem 1.3

In this section we are going to investigate the Hausdorff dimension of $\mathcal{U}_q^{(k)}$. First we show that $q_c \approx 2.32472$ is the critical base for $\mathcal{U}_q$.

Lemma 4.1. *Let $q > 1$. Then $\dim_H \mathcal{U}_q > 0$ if and only if $q > q_c$.*

Proof. The necessity follows by Theorem 1.2. Now we consider the sufficiency. Take $q \in (q_c, \infty)$. If $q > q^*$, then by Theorem 1.2 we have

$$\dim_H \mathcal{U}_q = \frac{\log q^*}{\log q} > 0.$$

Then it suffices to prove $\dim_H \mathcal{U}_q > 0$ for any $q \in (q_c, q^*]$.

Take $q \in (q_c, q^*]$. Recall from (3.3) that $\alpha(q_c) = q_c 1^\infty$ and $\alpha(q^*) = (q^*)^\infty$. Then by Lemma 2.3 it follows that there exists a sufficiently large integer $m \geq 1$ such that

$$\alpha(q) > q1^m q0^\infty.$$

Whence, by Lemma 2.2 one can verify that all sequences in

$$\Delta'_m := \prod_{i=1}^{\infty} \{q1^{m+1}, 1^{m+2}\}$$

excluding those ending with 1^∞ belong to $\mathcal{U}'_q$. This implies that

$$\dim_H \mathcal{U}_q \geq \dim_H \Delta_m(q), \tag{4.1}$$

where $\Delta_m(q) = \{((d_i))_q : (d_i) \in \Delta'_m\}$.

Note that $\Delta_m(q)$ is a self-similar set generated by the IFS

$$f_1(x) = \frac{x}{q^{m+2}} + (q1^{m+1}0^\infty)_q, \quad f_2(x) = \frac{x}{q^{m+2}} + (1^{m+2}0^\infty)_q,$$

which satisfies the open set condition (cf. [7]). Therefore, by (4.1) we conclude that

$$\dim_H \mathcal{U}_q \geq \dim_H \Delta_m(q) = \frac{\log 2}{(m+2)\log q} > 0.$$

□

In the following we will consider the Hausdorff dimension of $\mathcal{U}_q^{(k)}$ for any $k \geq 2$, and prove $\dim_H \mathcal{U}_q^{(k)} = \dim_H \mathcal{U}_q$. First we consider the upper bound of $\dim_H \mathcal{U}_q^{(k)}$.

Lemma 4.2. *Let $q > 1$. Then $\dim_H \mathcal{U}_q^{(k)} \leq \dim_H \mathcal{U}_q$ for any $k \geq 2$.*

Proof. Recall that $\phi_d(x) = (x+d)/q$ for $d \in \{0,1,q\}$. Then the lemma follows by observing that for any $k \geq 2$ we have

$$\mathcal{U}_q^{(k)} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{d_1\cdots d_n \in \{0,1,q\}^n} \phi_{d_1} \circ \cdots \circ \phi_{d_n}(\mathcal{U}_q).$$

□

Now we consider the lower bound of $\dim_H \mathcal{U}_q^{(k)}$. By Lemmas 4.1 and 4.2 it follows that

$$\dim_H \mathcal{U}_q^{(k)} = 0 = \dim_H \mathcal{U}_q$$

for any $q \le q_c$. So, in the following it suffices to consider $q > q_c$.

For $q > q_c$ let

$$F'_q(1) := \left\{(d_i) \in \mathcal{U}'_q : d_1 = 1\right\}$$

be the *follower set* in $\mathcal{U}'_q$ generated by the word 1, and let $F_q(1)$ be the set of $x \in E_q$ which have a q-expansion in $F'_q(1)$, i.e.,

$$F_q(1) = \left\{((d_i))_q : (d_i) \in F'_q(1)\right\}.$$

Lemma 4.3. *Let $q > q_c$. Then $\dim_H \mathcal{U}_q^{(k)} \ge \dim_H F_q(1)$ for any $k \ge 1$.*

Proof. For $k \ge 1$ and $q > q_c$ let

$$\Lambda_q^k := \left\{((d_i))_q : d_1 \cdots d_k = 0q^{k-1}, (d_{k+i}) \in F'_q(1)\right\}.$$

Then $\Lambda_q^k = \phi_0 \circ \phi_q^{k-1}(F_q(1))$, and therefore

$$\dim_H \Lambda_q^k = \dim_H F_q(1).$$

Hence, it suffices to prove $\Lambda_q^k \subseteq \mathcal{U}_q^{(k)}$.

Take

$$x_k = (0q^{k-1}(c_i))_q \in \Lambda_q^k \quad \text{with} \quad (c_i) \in F'_q(1).$$

We will prove by induction on k that x_k has k different q-expansions.

For $k = 1$, by Lemmas 2.1 and 2.2 it follows that $x_1 = (0(c_i))_q \in \mathcal{U}_q$. Suppose $x_k = (0q^{k-1}(c_i))_q$ has k different q-expansions. Now we consider x_{k+1}, which can be expanded as

$$x_{k+1} = (0q^k(c_i))_q = (10q^{k-1}(c_i))_q.$$

By Lemmas 2.1 and 2.2 we have $q^k(c_i) \in \mathcal{U}'_q$, and by the induction hypothesis it yields that $(0q^{k-1}(c_i))_q = x_k$ has k different q-expansions. This implies that x_{k+1} has at least $k+1$ different q-expansions. On the other hand, since $q > q_c > 2$, it gives that the union in (3.1) is disjoint. Then

$$x_{k+1} \in \phi_0(E_q) \cap \phi_1(E_q) \setminus \phi_q(E_q).$$

This implies that x_{k+1} indeed has $k+1$ different q-expansions, and we conclude that $\Lambda_q^k \subseteq \mathcal{U}_q^{(k)}$. $\square$

Lemma 4.4. *Let $q > q_c$. Then $\dim_H F_q(1) \ge \dim_H \mathcal{U}_q$.*

Proof. First we consider $q > q^*$. By Lemma 2.1 one can show that $\mathcal{U}'_q$ is contained in an irreducible sub-shift of finite type X'_A over the states $\{0,1,q\}$ with adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{4.2}$$

Moreover, the complement set $X'_A \setminus \mathcal{U}'_q$ contains all sequences ending with 1^∞. This implies that

$$\dim_H \mathcal{U}_q = \dim_H X_A(q), \tag{4.3}$$

where $X_A(q) := \{((d_i))_q : (d_i) \in X'_A\}$. Note that $X_A(q)$ is a graph-directed set satisfying the open set condition (cf. [23, Theorem 3.4]), and the sub-shift of finite type X'_A is irreducible. Then by (4.3) it follows that

$$\dim_H \mathcal{U}_q = \dim_H X_A(q) = \dim_H F_q(1).$$

Now we consider $q \in (q_c, q^*]$. Using Lemma 2.2, we have

$$\mathcal{U}'_q \subseteq \{0^\infty\} \cup \bigcup_{k=0}^{\infty} \{q^k 0^\infty\} \cup \bigcup_{k=0}^{\infty} \bigcup_{m=0}^{\infty} \{q^k 0^m F'_q(1)\},$$

where

$$q^k 0^m F'_q(1) := \left\{(d_i) : d_1 \cdots d_{k+m} = q^k 0^m, (d_{k+m+i}) \in F'_q(1)\right\}.$$

This implies that $\dim_H \mathcal{U}_q \le \dim_H F_q(1)$. $\square$

Proof of Theorem 1.3. The theorem follows directly by Lemmas 4.1–4.4. $\square$

5. Proof of Theorem 1.4

In the following we will consider the set $\mathcal{U}_q^{(\aleph_0)}$ which contains all $x \in E_q$ having countably infinitely many q-expansions.

Lemma 5.1. *For any $q \in \mathcal{B}_{\aleph_0}$ the set $\mathcal{U}_q^{(\aleph_0)}$ contains infinitely many points.*

Proof. It suffices to show that all of these points

$$z_k := (0^k q^\infty)_q, \quad k \geq 1,$$

are q-null infinite points, i.e., $z_k \in \mathcal{U}_q^{(\aleph_0)}$.

Note by Theorem 1.1 that $q \in \mathcal{B}_{\aleph_0} = [2, \infty)$. If $q > 2$, then by the proof of Lemma 3.3 it yields that $z_1 = (0q^\infty)_q$ is a q-null infinite point. Moreover, note that $z_k = \phi_0^{k-1}(z_1) \notin S_q$ for any $k \geq 2$. This implies that all of these points $z_k, k \geq 1$, are q-null infinite points. So, $\{z_k : k \geq 1\} \subseteq \mathcal{U}_q^{(\aleph_0)}$.

If $q = 2$, then by using the substitutions

$$0q \sim 10, \quad 0q^\infty = 1^\infty = q0^\infty,$$

one can also show that z_k is a q-null infinite point. Moreover, all of the q-expansions of $z_k = (0^k q^\infty)_q$ are of the form

$$0^k q^\infty, \quad 0^{k-1}1^\infty; \quad 0^{k-1}1^m 0 q^\infty, \quad 0^{k-1}1^{m-1} q 0^\infty,$$

where $m \geq 1$. Therefore, $z_k \in \mathcal{U}_q^{(\aleph_0)}$ for any $k \geq 1$. □

First we consider the set $\mathcal{U}_q^{(\aleph_0)}$ for $q \geq q^*$.

Lemma 5.2. *Let $q \geq q^*$. Then $\mathcal{U}_q^{(\aleph_0)}$ is at most countable.*

Proof. Let $x \in \mathcal{U}_q^{(\aleph_0)}$. Then x has a q-expansion (d_i) satisfying

$$|\Sigma(x_n)| = \infty$$

for infinitely many integers $n \geq 1$, where $x_n := ((d_{n+i}))_q$. This implies that (d_i) can not end in $\mathcal{U}'_q$.

Note by the proof of Lemma 4.4 that $\mathcal{U}'_q \subseteq X'_A$, where X'_A is a sub-shift of finite type over the state $\{0, 1, q\}$ with adjacency matrix A defined in (4.2). Moreover, $X'_A \setminus \mathcal{U}'_q$ is at most countable (cf. [23, Theorem 3.4]). Therefore, we will finish the proof by showing that the sequence (d_i) must end in X'_A.

Suppose on the contrary that (d_i) does not end in X'_A. Then the word $0q$ or 10 occurs infinitely many times in (d_i). Using the word substitution $0q \sim 10$ this implies that $x = ((d_i))_q$ has a continuum of q-expansions, leading to a contradiction with $x \in \mathcal{U}_q^{(\aleph_0)}$. □

Now we prove that $\mathcal{U}_q^{(\aleph_0)}$ is also countable for $q \in [2, q^*]$.

Lemma 5.3. *Let $q\in[2,q_c]$. Then $\mathcal{U}_q^{(\aleph_0)}$ is at most countable.*

Proof. Take $q\in[2,q_c]$. By Theorems 1.1 and 1.2 it follows that any $x\in E_q$ with $|\Sigma(x)|<\infty$ must belong to $\mathcal{U}_q=\{0,q/(q-1)\}$. Suppose $x\in\mathcal{U}_q^{(\aleph_0)}$. Then there exists a word $d_1\cdots d_n$ such that

$$\phi_{d_1}^{-1}\circ\cdots\circ\phi_{d_n}^{-1}(x)\in\mathcal{U}_q.$$

This implies that the set $\mathcal{U}_q^{(\aleph_0)}$ is at most countable, since

$$\mathcal{U}_q^{(\aleph_0)}\subseteq\bigcup_{n=1}^{\infty}\bigcup_{d_1\cdots d_n\in\{0,1,q\}^n}\phi_{d_1}\circ\cdots\circ\phi_{d_n}\left(\mathcal{U}_q\right).$$

□

In the following lemma we consider the Hausdorff dimension of the set $\mathcal{U}_q^{(\aleph_0)}$ for $q\in(q_c,q^*)$.

Lemma 5.4. *For $q\in(q_c,q^*)$ we have $\dim_H\mathcal{U}_q^{(\aleph_0)}\le\dim_H\mathcal{U}_q<1$.*

Proof. Take $q\in(q_c,q^*)$. Note that

$$\mathcal{U}_q^{(\aleph_0)}\subseteq\bigcup_{n=1}^{\infty}\bigcup_{d_1\cdots d_n\in\{0,1,q\}^n}\phi_{d_1}\circ\cdots\circ\phi_{d_n}(\mathcal{U}_q).$$

This implies that $\dim_H\mathcal{U}_q^{(\aleph_0)}\le\dim_H\mathcal{U}_q$. In the following it suffices to prove $\dim_H\mathcal{U}_q<1$.

Note that $\mathcal{U}_q'\subseteq X_A'$, where X_A' is the sub-shift of finite type over the state $\{0,1,q\}$ with adjacency matrix A defined in (4.2). Then

$$\mathcal{U}_q\subseteq X_A(q)=\{((d_i))_q:(d_i)\in X_A'\}.$$

Note that $X_A(q)$ is a graph-directed set (cf. [13]). This implies that

$$\dim_H\mathcal{U}_q\le\dim_H X_A(q)\le\frac{\log q_c}{\log q}<1.$$

□

In the following lemma we investigate the set $\mathcal{U}_q^{(2^{\aleph_0})}$ and show that $\mathcal{U}_q^{(2^{\aleph_0})}$ has full Hausdorff measure.

Lemma 5.5. *Let $q>1$. Then the set $\mathcal{U}_q^{(2^{\aleph_0})}$ has full Hausdorff measure, i.e.,*

$$\mathcal{H}^s(\mathcal{U}_q^{(2^{\aleph_0})})=\mathcal{H}^s(E_q)\in(0,\infty),$$

where $s=\dim_H E_q$.

Proof. Clearly, for $q\in(1,q^*]$ we have $E_q=[0,q/(q-1)]$, and therefore $s=\dim_H E_q=1$. Moreover, for $q>q^*$ we have by (1.1) that $s=\dim_H E_q=\log q^*/\log q$. Hence, the set E_q has positive and finite Hausdorff measure (cf. [14]), i.e.,

$$0<\mathcal{H}^s(E_q)<\infty \quad \text{for any} \quad q>1. \tag{5.1}$$

Moreover,

$$E_q=\mathcal{U}_q^{(2^{\aleph_0})}\cup\mathcal{U}_q^{(\aleph_0)}\cup\bigcup_{k=1}^\infty\mathcal{U}_q^{(k)}. \tag{5.2}$$

First we prove the lemma for $q\le q^*$. By Theorems 1.1 and 1.2 it follows that for any $q\in(1,q^*]$

$$\dim_H\mathcal{U}_q^{(k)}=\dim_H\mathcal{U}_q<1=\dim_H E_q \quad \text{for any} \quad k\ge 2.$$

Moreover, by Lemmas 5.2–5.4 we have

$$\dim_H\mathcal{U}_q^{(\aleph_0)}<1.$$

Therefore, by (5.1) and (5.2) we have $\mathcal{H}^s(\mathcal{U}_q^{(2^{\aleph_0})})=\mathcal{H}^s(E_q)\in(0,\infty)$.

Now we consider $q>q^*$. By Theorems 1.2, 1.3 and (1.1) it follows that

$$\dim_H\mathcal{U}_q^{(k)}=\frac{\log q_c}{\log q}<\frac{\log q^*}{\log q}=\dim_H E_q$$

for any $k=1,2,\cdots$. Moreover, by Lemma 5.2 we have $\dim_H\mathcal{U}_q^{(\aleph_0)}=0$. Therefore, the lemma follows by (5.1) and (5.2). $\square$

Proof of Theorem 1.4. The theorem follows by Lemmas 5.1–5.3 and 5.5. $\square$

6. Examples and final remarks

In the section we consider some examples. The first example is an application of Theorems 1.1–1.4 to expansions with deleted digits set.

Example 6.1. Let $q=3$. We consider q-expansions with digits set $\{0,1,3\}$. This is a special case of expansions with deleted digits (cf. [16]). Then

$$E_3=\left\{\sum_{i=1}^{\infty}\frac{d_i}{3^i}:d_i\in\{0,1,3\}\right\}.$$

By Theorems 1.2 and 1.3 we have

$$\dim_H\mathcal{U}_3^{(k)}=\dim_H\mathcal{U}_3=\frac{\log q_c}{\log 3}\approx 0.767877$$

for any $k\geq 2$. This means that the set $\mathcal{U}_3^{(k)}$ of $x\in E_3$ has k different expansions has the same Hausdorff dimension $\log q_c/\log 3$ for any integer $k\geq 1$.

Moreover, by Theorem 1.4 it yields that $\mathcal{U}_3^{(\aleph_0)}$ contains countably infinitely many points, and

$$\dim_H\mathcal{U}_3^{(2^{\aleph_0})}=\dim_H E_3=\frac{\log q^*}{\log 3}\approx 0.876036.$$

In terms of Theorem 1.2 we have a uniform formula of the Hausdorff dimension of $\mathcal{U}_q$ for $q\in[q^*,\infty)$. Excluding the trivial case for $q\in(1,q_c]$ that $\mathcal{U}_q=\{0,q/(q-1)\}$, it would be interesting to ask whether the Hausdorff dimension of $\mathcal{U}_q$ can be explicitly calculated for any $q\in(q_c,q^*)$.

In the following we give an example for which the Hausdorff dimension of $\mathcal{U}_q$ can be explicitly calculated.

Example 6.2. Let $q=1+\sqrt{2}\in(q_c,q^*)$. Then

$$(q0^\infty)_q=(1qq0^\infty)_q\quad\text{and}\quad\alpha(q)=(q1)^\infty.$$

Moreover, the quasi-greedy q-expansion of $q-1$ with alphabet $\{0,q-1,q\}$ is $q(q-1)^\infty$. Therefore, by Lemmas 3.1 and 3.2 of [23] it follows that

$\mathcal{U}_q'$ is the set of sequences $(d_i) \in \{0, 1, q\}^\infty$ satisfying

$$\begin{cases} d_{n+1}d_{n+2}\cdots < (1q)^\infty & \text{if} \quad d_n = 0, \\ 1^\infty < d_{n+1}d_{n+2}\cdots < (q1)^\infty & \text{if} \quad d_n = 1, \\ d_{n+1}d_{n+2}\cdots > 01^\infty & \text{if} \quad d_n = q. \end{cases}$$

Let X_A' be the sub-shift of finite type over the states

$$\{00, 01, 11, 1q, q0, q1, qq\}$$

with adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then one can verify that $\mathcal{U}_q' \subseteq X_A'$, and $X_A' \setminus \mathcal{U}_q'$ contains all sequences ending with 1^∞ or $(1q)^\infty$. This implies that

$$\dim_H \mathcal{U}_q = \dim_H X_A(q),$$

where $X_A(q) = \{((d_i))_q : (d_i) \in X_A'\}$.

Note that $X_A(q)$ is a graph-directed set satisfying the open set condition (cf. [13]). Then by Theorem 1.3 we have

$$\dim_H \mathcal{U}_q^{(k)} = \dim_H \mathcal{U}_q = \frac{h(X_A')}{\log q} \approx 0.691404.$$

Furthermore, by the word substitution $q00 \sim 1qq$ and in a similar way as in the proof of Lemma 5.2 one can show that $\mathcal{U}_q^{(\aleph_0)}$ contains countably infinitely many points.

Finally, by Theorem 1.4 we have $\dim_H \mathcal{U}_q^{(2^{\aleph_0})} = \dim_H E_q = 1$.

Question 1. Can we give a uniform formula for the Hausdorff dimension of $\mathcal{U}_q$ for $q \in (q_c, q^*)$?

In beta expansions we know that the dimension function of the univoque set has a devil's staircase behavior (cf. [11]).

Question 2. Does the dimension function $D(q) := \dim_H \mathcal{U}_q$ has a devil's staircase behavior in the interval (q_c, q^*)?

By Theorem 1.4 it gives that $\mathcal{U}_q^{(\aleph_0)}$ is countable for any $q \in \mathcal{B}_2 \setminus (q_c, q^*)$. Moreover, in Lemma 5.4 we show that $\dim_H \mathcal{U}_q^{(\aleph_0)} \leq \dim_H \mathcal{U}_q < 1$ for any $q \in (q_c, q^*)$. In terms of Example 6.2 we made the following question.

Question 3. Does the set $\mathcal{U}_q^{(\aleph_0)}$ have positive Hausdorff dimension for $q \in (q_c, q^*)$?

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